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ON INVERSION OF THE LAGRANGE-DIRICHLET THEOREM*

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It has been shown /1/ that if the potential energy of a mechanical system is an analytic function of the form

$$V(\mathbf{q}) = V_m(\mathbf{q}) + V_{m+1}(\mathbf{q}) + \ldots \quad (m \geqslant 2)$$

where V_i are polynomials of degree i, then the equilibrium position $q = 0$ is unstable for odd m. Instability has also been proved for even m /2/ on the assumption that $\mathbf{q} = 0$ is not a local minimum point of $V_m(\mathbf{q})$. In this paper, with suitable additional assumptions, using the procedure employed in /2/, we shall prove instability in some cases when m is even and $V_m(\mathbf{q})$ is positive definite.

Consider a mechanical system with n degrees of freedom described by a Lagrange function

$$L(\mathbf{q}, \mathbf{q}^{\cdot}) = {}^1/{}_2\mathbf{q}^{\cdot T} A(\mathbf{q})\mathbf{q}^{\cdot} - V(\mathbf{q}), \quad \mathbf{q} \in R^n \tag{1}$$

where A is a symmetric positive-definite $n \times n$ matrix and V is the potential energy. The equations of motion are

$$A(\mathbf{q})\mathbf{q}^{\cdot\cdot} + \left(\frac{d}{dt}A(\mathbf{q})\right)\mathbf{q}^{\cdot} - \frac{\partial}{\partial \mathbf{q}}({}^1/{}_2\mathbf{q}^{\cdot T} A(\mathbf{q})\mathbf{q}^{\cdot}) = -V'(\mathbf{q}) \tag{2}$$

The Lagrange-Dirichlet Theorem states that the equilibrium position $\mathbf{q} = 0$ is stable if $V(0) = 0$ is a strict local minimum of V /3/.

The immediate inversion of this theorem is false, as shown in /4/ for a system with one degree of freedom having $V(0) = 0$, $V(q) = \exp(-1/q^2)\cos(1/q)$ if $q \neq 0$. Letting B_ε denote an open sphere of radius ε, we notice that this potential has the following property: for any $\varepsilon > 0$ there exists a connected open set $G: 0 \in G \subseteq \bar{G} \subseteq B_\varepsilon$ such that $V(q) > 0$ on ∂G. Even this property, which is weaker than the positive definiteness of V, is not necessary for stability in R^n /5/.

There remains the assumption that the equilibrium is unstable if V is analytic and does not have a minimum at the origin. Numerous results of this kind have been established; references to most of them may be found in /6-8/.

Without claiming to provide a complete list, we will recall some of these results. Let

$$V(\mathbf{q}) = V_m(\mathbf{q}) + V_{m+1}(\mathbf{q}) + \ldots \quad (m \geqslant 2)$$

where V_i are homogeneous polynomials of degree i. If $V_m(\mathbf{q})$ is negative definite or if $m = 2$ and $V_2(\mathbf{q})$ may take negative values, the inversion was established by Lyapunov /9/. If $V_2(\mathbf{q}) \geqslant 0$ and only one of the Poincaré coefficients vanishes, the inversion was proved by Koiter /10/. Some inversion results have been based on the following fundamental lemma of Chetayev /11/, which guarantees instability if

1) $\theta = \{\mathbf{q} : V(\mathbf{q}) < 0\} \neq \varnothing$, $0 \in \partial\theta$

2) $(\mathbf{q} \mid \partial V/\partial \mathbf{q}) < 0$ on $\partial\theta$

As an immediate corollary of this lemma one can prove instability in the case when $V = V_m(\mathbf{q})$ is homogeneous or there exists $k \geqslant 2$ such that $V_i(\mathbf{q}) \geqslant 0$ for $i < k$ and $V_i(\mathbf{q}) \leqslant 0$ for $k > i$ /11/. Condition 2 is not always satisfied in reality, even if V has a strict local maximum at $\mathbf{q} = 0$. This case was studied in /7/ for $V \in C^2$ /6/ and for functions $V \in C^1$ such that $V(0)$ is a non-strict local maximum /12/.

We note that Chetayev's lemma was strengthened in /5/, where condition 2 was replaced by

3) $(\mathbf{q} \mid \partial V/\partial \mathbf{q}) + \alpha V(\mathbf{q}) < 0$ on θ, $0 < \alpha < 2$

and also by Chetayev himself /11/, who used a vector field $\mathbf{f}(\mathbf{q})$ of C^1 smoothness on θ in

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R^n such that $f(0)=0$, $A(0)\frac{\partial f}{\partial q}(0)$ is a positive definite matrix, with the condition

4) $(f \mid \partial V/\partial q) < 0$ on θ

instead of condition 2.

This condition will be used quite frequently in what follows. It was used in /3/, where the inversion theorem in its entirety was proved for two degrees of freedom, on the assumption that $A(0)$ is the identity matrix and $V(q)$ is an analytic function which takes negative values in any neighbourhood of the origin. The same result was obtained independently in /14/, while the condition $A(0)=I$ was eliminated in /15/. For $n=2$ and an analytic function V with a non-isolated minimum at $q=0$, an attempt to prove the theorem was made in /16/.

Instability was proved for the case of arbitrary n and a non-analytic potential in /13/, under the following assumptions:

$A(q)=I$, where I is the $n\times n$ identity matrix.

$$V(q)=V_m(q)+R(q),\ m\geqslant 2,\ R'(0)=\ldots=R^{(m)}(0)=0$$

The function $V_m(q)$ is non-degenerate.

Instability was established in /17/ on the assumption that V is quasihomogeneous or semi-quasihomogeneous and $A(0)=I$. The first of these results extends a result of /11/ for $V=V_m$, the second generalizes the aforementioned result of /13/ to the case in which V is analytic. In /18/ the condition $A(0)=I$ was eliminated under an additional assumption and the result of /13/ was slightly generalized.

Instability was established in /1/ under rather simple assumptions: $m>2$ and is odd. Finally, instability was proved in /2/ for arbitrary $m\geqslant 2$ provided only that $V_m(q)$ can take negative values, i.e., without requiring that V_m be non-degenerate. This remarkable result almost settles the inversion question for analytic potentials. The only remaining open question is: what happens if $V(q)=V_{2p}(q)+\ldots$, with $V_{2p}(q)\geqslant 0$ and $V(0)$ not a local minimum? Instability was proved in /19/ on the assumption that $V=V_2+V_m+\ldots, V_2\geqslant 0$ and V_m takes strictly negative values on the hypersurface $\{V_2(q)=0\}$. In the sequel, using exactly the same procedure as in /2/, we shall obtain some instability results in the case $V=V_{2p}+\ldots$, $p>1$.

Retaining the previous notation, let us assume that $A(q)$ and $V(q)$ are analytic and

$$A(q)=I+\varphi(q),\quad \varphi(0)=0,\quad V(q)=V_{2m}(q)+V_{2m+2}(q)+\ldots,\quad m\geqslant 3$$

where V_k is a homogeneous polynomial of degree k, D^rV_k its partial derivative of order r.

Theorem 1. If the following conditions hold:

1° $V_{2m}(q)\geqslant 0$ in R^n, $S=\{q\colon V_{2m}(q)=0\}$;

2° $\exists e\in S$, $\|e\|=1$, $V_{2m+2}(e)=\min\{V_{2m+2}(q)\colon \|q\|=1\}<0$;

3° $D^rV_{2m}(e)=0$ for $r=2,3,4$;

4° $\varphi(q)/\|q\|^2\to 0$ as $q\to 0$,

then the equlibrium of Eqs.(2) at $q=0$ is unstable.

Proof. By condition 4°, Eqs.(2) may be written in the form

$$q^{\cdot\cdot}+G(q,q^{\cdot})+v(q)=0 \tag{3}$$

$$v(q)=V'_{2m}(q)+V'_{2m+2}(q)+\sum_{k\geqslant 2m+2}v_k(q)$$

where G is an analytic and quadratic function of $q^{\cdot}$ and v_k are homogeneous polynomials of degree k. We can now follow the usual procedure /2/. Let F be the space of the formal series

$$q(t)=\sum_{mj\leqslant i}a_{ij}(\ln(t))\,t^{-i/m},\quad a_{ij}\in R^n,\quad i,j=0,1,\ldots$$

Let $t^{\nu}F$ denote the space of the series $t^{\nu}q$, $q\in F$. We are going to construct a formal solution $q\in t^{-1/m}F$ of Eqs.(3), determining the coefficients a_{ij} by induction - forward induction on j and backward on i. The induction process begins from the first term

$$q_1(t)=a_{00}t^{-1/m}$$

Substituting $q=q_1(t)$ on the left of Eqs.(3), we obtain a series

$$V'_{2m}(a_{00})\,t^{-2+1/m}+[(1/m)(1/m+1)\,a_{00}+V'_{2m+2}(a_{00})]\,t^{-2-1/m}+\ldots$$

(the dots stand for terms of degree t strictly less than $-2-1/m$). The first two coefficients of this series vanish if one takes $a_{00}=ae$, where a is a suitable real number and e is defined as in condition 2°. Indeed, since $e\in S$, $V_{2m}(ae)=0$ is the absolute minimum of V_{2m}

and so $V'_{2m}(a\mathbf{e}) = 0$. The second coefficient vanishes if

$$(1/m)(1/m+1)a\mathbf{e} - V'_{2m+2}(\mathbf{e})a^{2m+1} = 0$$

By condition 2° there exists a real number $c > 0$ such that $V'_{2m+2}(\mathbf{e}) = -c\mathbf{e}$. Consequently, it will suffice to take

$$a = ((m+1)/(cm^2))^{1/2m}$$

Now, putting $a_{00} = a\mathbf{e}$ and

$$\mathbf{y}_2 = \mathbf{q}_1^{\cdot\cdot} + \mathbf{G}(\mathbf{q}_1, \mathbf{q}_1^{\cdot}) + \mathbf{v}(\mathbf{q}_1)$$

we obtain

$$\mathbf{y}_2 \in t^{-2-1/m}F$$

and $\mathbf{y}_2$ contains no terms of degree t higher than $-2-2/m$. Now assume for the induction step that for some integer $N \geqslant 2$ we have found a series

$$\mathbf{q}_{N-1} = \sum_{mj \leqslant i < N-1} a_{ij}(\ln t)^j t^{-i-1/m}$$

such that

$$\mathbf{y}_N = \ddot{\mathbf{q}}_{N-1} + \mathbf{G}(\mathbf{q}_{N-1}, \dot{\mathbf{q}}_{N-1}) + \mathbf{v}(\mathbf{q}_{N-1})$$

lies in the space $t^{-2-1/m}F$ and contains no powers of t higher than $-2-N/m$. Our problem now is to construct a series such that y_{N+1} lies in the space $t^{-2-1/m}F$ and contains no powers of t higher than $-2-(N+1)/m$. To that end we put

$$\mathbf{q}_N = \mathbf{q}_{N-1} + \Delta\mathbf{q}, \quad \Delta\mathbf{q} = \sum_{mj \leqslant N-1} a_{N-1,\,j}(\ln t)^j t^{-N/m}$$

Then

$$\begin{aligned}\mathbf{y}_{N+1} = \mathbf{y}_N + \Delta\mathbf{q}^{\cdot\cdot} + [V'_{2m}(\mathbf{q}_N) - V'_{2m}(\mathbf{q}_{N-1})] + [V'_{2m+2}(\mathbf{q}_N) - V'_{2m+2}(\mathbf{q}_{N-1})] + \\ [\mathbf{G}(\mathbf{q}_N, \mathbf{q}_N^{\cdot}) - \mathbf{G}(\mathbf{q}_{N-1}, \dot{\mathbf{q}}_{N-1})] + \sum_{k \geqslant 2m+2} [\mathbf{v}_k(\mathbf{q}_N) - \mathbf{v}_k(\mathbf{q}_{N-1})]\end{aligned} \tag{4}$$

The right-hand side of (4) is exactly the same as in /2/, except for the expression in the first pair of brackets. Since $\mathbf{G}$ is quadratic in $\mathbf{q}^{\cdot}$ and $\mathbf{v}_k$ are homogeneous polynomials of degree k, it is obvious from the Taylor expansion at the point $(\mathbf{q}_{N-1}, \dot{\mathbf{q}}_{N-1})$ that the last two expressions in brackets in (4) contain no powers of t higher than $-2-(N+1)/m$.

The expression in the second pair of brackets in (4) may be rewritten in the form

$$V''_{2m+2}(\mathbf{q}_{N-1})\,\Delta\mathbf{q} + \ldots = V''_{2m+2}(\mathbf{q}_1)\,\Delta\mathbf{q} + \ldots$$

where the dots stand for terms of degree t strictly less than $-2-N/m$ and the degree of $V''_{2m+2}(\mathbf{q})$ is $-2-N/m$.

We will now take a closer look at the expression in the first pair of brackets, which does not appear in /2/. Expanding it in Taylor series in the neighbourhood of $\mathbf{q}_{N-1}$ and neglecting high-order terms as usual, we express it as

$$D^2V_{2m}(\mathbf{q}_{N-1})\,\Delta\mathbf{q} + {}^1/_2 D^3V_{2m}(\mathbf{q}_{N-1})\,\Delta\mathbf{q}^2 + {}^1/_6 D^4V_{2m}(\mathbf{q}_{N-1})(\Delta\mathbf{q})^3 + \ldots, \tag{5}$$

where the dots stand for powers not exceeding $-2+5/m-4N/m$, hence not exceeding $-2-(N+1)/m$. If $N=2$ the terms written out in (5) vanish by condition 3°. If $N>2$, we expand $D^kV_{2m}(\mathbf{q}_{N-1})$ $(k=2,3,4)$ in Taylor series in the neighbourhood of $\mathbf{q}_1$. Using condition 3°, we obtain, to within the same higher-order terms,

$$\begin{aligned}D^2V_{2m}(\mathbf{q}_{N-1}) &= {}^1/_6 D^5V_{2m}(\mathbf{q}_1)(\mathbf{q}_{N-1} - \mathbf{q}_1)^3 + \ldots \\ D^3V_{2m}(\mathbf{q}_{N-1}) &= {}^1/_2 D^5V_{2m}(\mathbf{q}_1)(\mathbf{q}_{N-1}) - \mathbf{q}_1)^2 + \ldots \\ D^4V_{2m}(\mathbf{q}_{N-1}) &= D^5V_{2m}(\mathbf{q}_1)(\mathbf{q}_{N-1} - \mathbf{q}_1) + \ldots\end{aligned}$$

Since $D^5V_{2m}(\mathbf{q}_1)$ contains powers of t with exponent of at most $-2+5/m$, and since the degrees of $(\mathbf{q}_{N-1} - \mathbf{q}_1)$ are at most $-2/m$, the highest power of t appearing in D^kV_{2m} is $-2-1/m$, $-2+1/m$, $-2+3/m$ for $k=2,3,4$, respectively. But $\Delta\mathbf{q}$ has degree $-N/m$ and $N>2$. Thus, all the terms in (5), and hence all the terms in the first bracketed expression in (4), have a degree in t of at most $-2-(N+1)/m$.

By the induction hypothesis $\mathbf{y}_N$ does not contain powers of t higher than $-2-N/m$. Letting $\mathbf{z}$ denote the sum of terms including $t^{-2-N/m}$ in $\mathbf{y}_N$, we exclude all terms of degree $-2-N/m$ in $\mathbf{y}_{N+1}$, solving the equation

$$\Delta\mathbf{q}^{\cdot\cdot} + V''_{2m+2}(\mathbf{q}_1)\,\Delta\mathbf{q} = -\mathbf{z} = \sum_{mj \leqslant N-1} \mathbf{z}_j(\ln t)\,t^{-2-N/m} \tag{6}$$

This equation is the same one as in /2/, and all arguments presented there are valid for (6). For completeness, we will complete the construction of the formal solution, repeating those arguments. The highest degree of the logarithm on the right of (6) is equal to the integer part of $(N-1)/m$, i.e., $M = [(N-1)/m]$. Collecting the terms of this degree we obtain

$$(N/m)\,(N/m+1)\,a_{N-1,M} + V''_{2m+2}(a\mathbf{e})\,a_{N-1,m} = \mathbf{z}_m \tag{7}$$

This equation has a unique solution $a_{N-1,m}$, provided that $(-N/m)\cdot(N/m+1)$ is not an eigenvalue of $V''_{2m+2}(a\mathbf{e})$. The tensor $V''_{2m+2}(a\mathbf{e})$ is symmetric and all its eigenvalues are real.

The vector $\mathbf{e}$ is an eigenvector with eigenvalue $-((m+1)/m)\,(2m+1)/m$. Indeed, by Euler's Theorem

$$V''_{2m+2}(a\mathbf{e})\,\mathbf{e} = \frac{2m+1}{a}\,V'_{2m+2}(a\mathbf{e}) = -\frac{2m+1}{m}\,\frac{m+1}{m}\,\mathbf{e}$$

The space orthogonal to $\mathbf{e}$ is tangent to the unit sphere at $\mathbf{e}$. Since $V_{2m+2}(\mathbf{e})$ is a minimum on the unit sphere, all the eigenvalues corresponding to this space are non-negative. Thus Eq.(7) has a unique solution if $N \neq m+1$.

If $N \neq m+1$, we substitute $\Delta\mathbf{q}$ into Eq.(6) with the coefficient $a_{N-1,m}$ obtained from Eq.(7), the other coefficients $a_{N-1,j}$ remaining undetermined. Then the terms involving $(\ln t)^M$ will vanish in Eq.(6). To eliminate the terms involving $(\ln t)^{M-1}$, we solve the equation

$$(N/m)\,(N/m+1)\,a_{N-1,M-1} + V''_{2m+2}(a\mathbf{e})\,a_{N-1,M-1} = \mathbf{z}^*_{M-1}$$

where z^*_{M-1} is a known quantity which depends on z_{M-1} and $a_{N-1,M}$. An $(M+1)$-fold repetition of the same procedure will solve Eq.(6).

We will now consider the exceptional case $N = m+1$, $M = 1$, when

$$\mathbf{q}_m = \sum_{i<m} a_{i0}t^{-(i+1)/m}, \quad \Delta\mathbf{q} = (a_{m0} + a_{m1}\ln t)\,t^{-(1+1/m)}$$

Neither $\mathbf{y}_{m+1}$ nor the right-hand side of Eq.(6) contains terms involving logarithms. Thus the equation may be written in the form

$$\Delta\mathbf{q}'' + V''_{2m+2}(\mathbf{q}_1)\,\Delta\mathbf{q} = (\lambda\mathbf{e} + \mathbf{f})\,t^{-3-1/m} \tag{8}$$

where λ is a real number and $\mathbf{f}$ a vector orthogonal to $\mathbf{e}$. We will split (8) into equations for the components $\mathbf{e}$ and $\mathbf{f}$. Since all the eigenvalues of $V''(a\mathbf{e})$ corresponding to a space orthogonal to $\mathbf{e}$ are non-negative, the equation

$$(1+1/m)\,(2+1/m)\,a_{m0}t^{-3-1/m} + V''_{2m+2}(\mathbf{q}_1)\,a_{m0}t^{-1-1/m} = \mathbf{f}t^{-3-1/m}$$

has a unique solution a_{m0} which is orthogonal to $\mathbf{e}$. The remaining equation

$$-(3+2/m)\,a_{m1} + [(1+1/m)\,(2+1/m) + V''_{2m+2}(a\mathbf{e})]\ln t\,a_{m1} = \lambda\mathbf{e}$$

holds for $a_{m1} = -\lambda\mathbf{e}/(3+2/m)$. This completes the construction of the formal solution.

It was shown in detail in /2/ that the formal solution converges to a solution of Eq.(3). The only difference in the present case is the extra term $V_{2m}(\mathbf{q})$ in the potential. Most of the algebra in /2/ are not affected by the introduction of this additional term and may be represented here without any alteration; we will therefore omit them.

Only one argument in /2/ needs some attention because of the extra term V_{2m}. The formula

$$t^2\,(V''_{2m+2}(\mathbf{q}_1) - \mathbf{v}'(\mathbf{q}_{2m})) = V''_{2m+2}(a_{00}) - V''_{2m+2}(t^{1/m}\mathbf{q}_{2m}) - t^{-1/m}\mathbf{v}'_{2m+2}(t^{1/m}\mathbf{q}_{2m}) + \ldots \tag{9}$$

where the dots stand for powers of t with exponents less than $-1/m$, yields an estimate $O(t^{-1/m} + t^{-1}\ln t)$ for the norm of the operator in (9) for large t.

We shall show that the addition of V_{2m} to the potential does not change this estimate. Clearly,

$$t^2 V''_{2m}(\mathbf{q}_{2m}) = t^{2/m} V_{2m}(t^{1/m}\mathbf{q}_{2m})$$

and

$$t^{1/m}q_{2m} = a_{00} + a_{10}t^{-1/m} + \ldots + a_{2m-1,0}t^{-2+1/m} + a_{m,1}t^{-1}\ln t + \ldots + a_{2m-1,1}t^{-2+1/m}\ln t$$

By condition 3°, the Taylor expansion of V_{2m} at a_{00} begins with terms of degree 3 in $t^{-1/m}$ and $t^{-1}\ln t$. Thus, for large t,

$$t^2 V''_{2m}(\mathbf{q}_{2m}) = O(t^{-1/m} + t^{-1}\ln t)$$

and we obtain the same estimate for the norm of the operator in (9) as in /2/.

Thus the solution we have constructed converges and the instability of the origin follows from the existence of an asymptotic solution.

Remark 1. Theorem 1 generalizes the results obtained in /2/ in the case $\mathbf{q} \in R^n$. Indeed, if $V_{2m} \equiv 0$, the conditions 1° and 3° are automatically satisfied, 4° is unnecessary and 2° follows from the previous assumptions in /2/.

Remark 2. The applicability of Theorem 1 is of course limited by the fact that condition 3° for the non-zero function V_{2n} requires $m \geqslant 3$. In addition, the method employed to prove the theorem does not work if the minimum of V_{2m+2} on the unit sphere lies outside the domain $V_{2m}(q) > 0$. This becomes clear if one considers the positive definite potential

$$V = x^8 - 4x^4y^6 + 5y^{12}$$

and the potential

$$V = x^8 - 4x^4y^6 + y^{12}$$

which makes the equilibrium position unstable (see /1/).

The method is not applicable when $V'_{2m}(\mathrm{e})\,\mathrm{e} = 2mV_{2m}(\mathrm{e}) \neq 0$ and the condition $V_{2m}(\mathrm{e}) = 0$, which appears in the proof, fails to hold. We note that the condition $V'_{2m}(\mathrm{e}) = 0$ here is equivalent to the condition $V_{2m}(\mathrm{e}) = 0$ or $\mathrm{e} \in S$.

Theorem 1 may be applied if $m \geqslant 3$ and the variables appearing in V_{2m+2} differ from those appearing in V_{2m}, e.g., if

$$V(x, y, z, u) = x^6 + y^8 + z^8 - u^8 + \sum_{k>8} V_k(x, y, z, u)$$

Appendix 1. Recalling the procedure used to prove Theorem 1, we can extend that theorem to the case $V(\mathbf{q}) = V_{2k}(\mathbf{q}) + \ldots + V_{2m+1}(\mathbf{q}) + V_{2m+2}(\mathbf{q}) + \ldots$, replacing conditions 1°-4° by the following alternatives:

5° $V_{2m+2}(\mathbf{q})$ has a strictly negative local minimum on the unit sphere;

6° $D^jV_i(\mathrm{e}) = 0,\ 2k \leqslant i \leqslant 2m+1,\ 1 \leqslant j \leqslant 4 + 2m - i$;

7° $\varphi(\mathbf{q}) / |\mathbf{q}|^{2m-2k+2} \to 0$ as $\mathbf{q} \to 0$.

Note that it follows from the condition $D^1V_i(\mathrm{e}) = 0$ that e lies in the sets $V_i(\mathbf{q}) = 0$ $(i = 2k, \ldots, 2m+1)$. These conditions are not equivalent to those of Theorem 1, where $k = m$. We note, moreover, that the functions $V_{2m}, V_{2k} + V_{2k+1}, \ldots, V_{2k} + \ldots + V_{2m+1}$ are not necessarily always positive.

Appendix 2. A slight modification, analogous to that outlined above, makes Theorem 1 applicable to the case

$$V(\mathbf{q}) = V_{2k}(\mathbf{q}) + \ldots + V_{2m}(\mathbf{q}) + V_{2m+1}(\mathbf{q}) + \ldots$$

when the negative part of the potential is determined by a term of odd degree $(2m+1)$. This is done by adopting the following assumptions instead of 1°-4°:

8° $V_{2m+1}(\mathbf{q})$ has a strictly negative local minimum on the unit sphere;

9° $D^jV_i(\mathrm{e}) = 0,\ 2k \leqslant i \leqslant 2m,\ 1 \leqslant j \leqslant 3 + 2m - i$;

10° $\varphi(\mathbf{q})/|\mathbf{q}|^{2m-2k+1} \to 0$ as $\mathbf{q} \to 0$.

The proof is very similar to that presented above and need not be reproduced in full. The first approximation to the solution of Eq.(1) will be

$$\mathbf{q}_1(t) = a_0 t^{-\mu},\quad a_0 \in R^n,\quad \mu = 2/(2m-1)$$

$$a_0 = a\mathrm{e},\quad a^{2m-1}c = 2(2m+1)(2m-1)^{-2},\quad V'_{2m+1}(\mathrm{e}) = -c\mathrm{e}$$

Consider the space of formal series

$$F = \{\mathbf{q}(t) = \sum a_i t^{-i\mu},\ a_i \in R^n,\ i = 0, 1, \ldots\}$$

and assume that for $N \geqslant 2$ we have found an $(N-1)$-th approximation

$$\mathbf{q}_{N-1}(t) = t^{-\mu}(a_0 + a_1 t^{-\mu} + \ldots + a_{N-2}t^{-(N-2)\mu})$$

in $t^{-\mu}F$ such that

$$\mathbf{y}_N = \ddot{\mathbf{q}}_{N-1} + \mathbf{G}(\mathbf{q}_{N-1}, \dot{\mathbf{q}}_{N-1}) + \mathbf{v}(\mathbf{q}_{N-1})$$

is a member of $t^{-2-\mu}F$ and involves no powers of t with exponent greater than $-2 - N\mu$. Then we choose

$$\mathbf{q}_N = \mathbf{q}_{N-1} + \Delta\mathbf{q},\quad \Delta\mathbf{q} = a_{N-1}t^{-N\mu}$$

such that $\mathbf{y}_{N+1}$ is in $t^{-2-\mu}F$ and involves no powers of t with exponent greater than $-2-(N+1)\mu$.

To complete the proof we need only solve the equation

$$N\mu(N\mu+1)\,a_{N-1} + V''_{2m+1}(a\mathrm{e})\,a_{N-1} = \mathbf{z} \tag{10}$$

where z is a term of degree $-2-N\mu$ in y_N. The operator $V''_{2m+1}(ae)$ has exactly one negative eigenvalue corresponding to the eigenvector e; it is $(-4m)(2m+1)(2m-1)^{-2}$. It can be verified that for any positive integer N the number $(-N\mu)(N\mu+1)$ is not an eigenvalue of $V''_{2m+1}(ae)$. Consequently, Eq.(10) has a unique solution for any positive integer N and the formal solution can be constructed in $t^{-\mu}F$.

To show that this formal solution in fact converges, we repeat the arguments of /2/, replacing the quantities m and $p=m/2-1$ appearing in /2/ by $2m+1$ and $m-{}^1/_2=1/\mu$, respectively, and omitting the logarithms throughout. Condition 9° implies the continuing validity of the estimates in /2/ despite the extra term $(V_{2k}+\ldots+V_{2m})$ in the potential. This completes the proof.

Comparing the reasoning with Theorem 1, we observe that the formal solution in Appendix 2 does not contain logarithmic terms. This simplification is possible because there are no critical values of N in Eq.(10), whereas Eq.(7) of Theorem 1, unlike (10), has a critical value $N=m+1$, which generates logarithmic terms.

Finally, we note that if $V_{2k}=\ldots=V_{2m}\not\equiv 0$, condition 10° may be dropped and condition is automatically valid.

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